5.1: The Natural Logarithmic Function: Differentiation

When you are done with your homework you should be able to…

- π Develop and use properties of the natural logarithmic function
- π Understand the definition of the number e
- π Find derivatives involving the natural logarithmic function

Warm-up:

1. Use the limit definition of the derivative to find the derivative of $f(x)=\frac{3}{x}$ with respect to x.

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$f(x)=\frac{3}{x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{\frac{3}{(x+\Delta x)}\cdot\frac{x}{x}-\frac{3}{x}\cdot\frac{(x+\Delta x)}{(x+\Delta x)}}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{3x-3x-3\Delta x}{\Delta x(x)(x+\Delta x)}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{-3\Delta x}{\Delta x(x)(x+\Delta x)}$$

D.S.

$$f'(x)=\frac{-3}{x(x+0)}$$

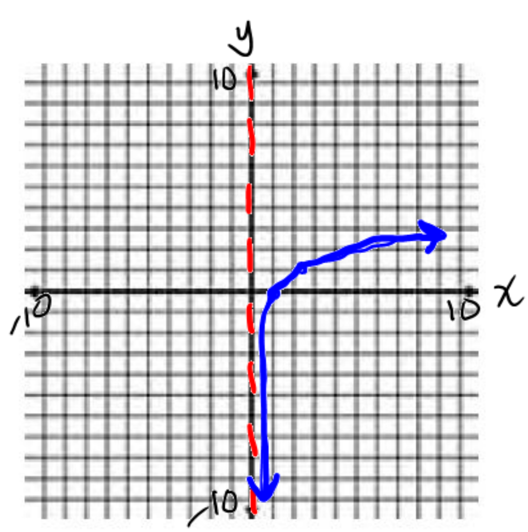
$$f'(x)=-\frac{3}{x^2}$$

2. Graph $y=\ln_e x$. $\leftrightarrow e^y=x$

$$\lim_{x\to\infty}\ln x=\infty \text{ (DNE)}$$

Finite

$$\lim_{x\to 0^+}\ln x=-\infty \text{ (DNE)}$$



DEFINITION OF THE LOGARITHMIC FUNCTION BASE e

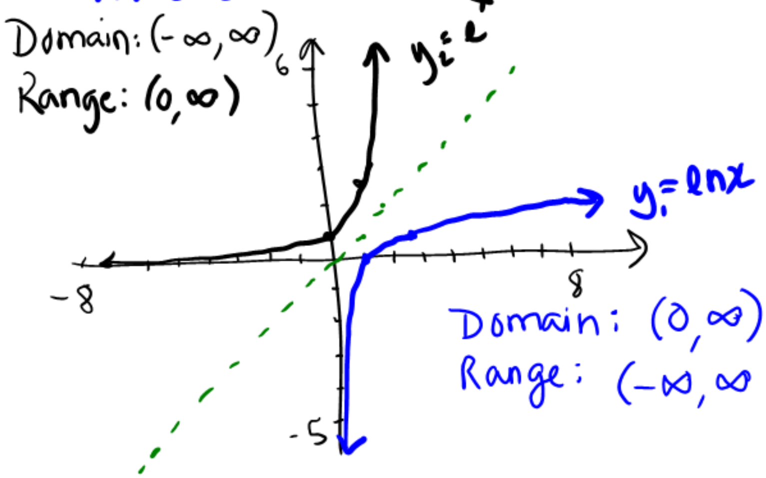
If a is a positive real number ($a \neq 1$) and x is any positive real number, then the **logarithmic function to the base e** is defined as

$\log_a x =$ $\log_e x = \ln x$.

The number e:

The number e can be defined as a limit; specifically, $\lim_{v \to 0}(1+v)^{1/v}$.

The natural logarithmic function and the natural exponential function are inverses of each other.



iff = if and only if

$\ln x = \log_e x$

PROPERTIES OF INVERSE FUNCTIONS

1. $y = e^x$ iff $\log_e y = x$ ($\ln y = x$)
2. $e^{\ln x} = x$, for $x > 0$
3. $\ln e^x = x$, for all x

PROPERTIES OF NATURAL LOGS

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln xy = \ln x + \ln y$
4. $\ln \frac{x}{y} = \ln x - \ln y$
5. $\ln x^n = n \ln x$

The inverse relationship between the natural logarithmic function and the natural exponential function can summarized as follows:

$$\ln e^{x} = x \quad \text{and} \quad e^{\ln x} = x$$

Example 1: Condense the following logarithmic expressions.

a. $\ln(x+8)-[\ln(x-2)-5\ln(x)]$

$= \ln(x+8)-[\ln(x-2)-\ln x^5]$

$= \ln(x+8)-\left[\ln \frac{x-2}{x^5}\right]$

$= \ln \frac{x+8}{\left(\frac{x-2}{x^5}\right)}$

$= \ln \frac{x^5(x+8)}{x-2}$

b. $\frac{1}{2}\ln x + 8\ln z - \ln y$

$= \ln x^{1/2} + \ln z^8 - \ln y$

$= \ln x^{1/2} z^8 - \ln y$

$= \ln \frac{x^{1/2} z^8}{y}$

Example 2: Expand the following logarithmic expressions.

a. $\ln \sqrt[4]{\left(\frac{x^2-1}{2x+5}\right)^3}$ $= \ln\left(\frac{x^2-1}{2x+5}\right)^{3/4}$

$= \frac{3}{4}\ln\frac{x^2-1}{2x+5}$

$= \frac{3}{4}\left(\ln(x^2-1) - \ln(2x+5)\right)$

b. ~~$\left(\frac{1-\cos x}{\cos 2x}\right)^5$~~ correction

$\ln\left(\frac{1-\cos x}{\cos 2x}\right)^5 = 5\ln\left(\frac{1-\cos x}{\cos 2x}\right)$

$= 5\left[\ln(1-\cos x) - \ln(\cos 2x)\right]$

Example 3: Solve the following equations. Give the **exact result** and then round to 3 decimal places.

a. $\ln(x-2)+\ln(x+2)=16$

$\ln[(x-2)(x+2)]=16$

$\ln(x^2-4)=16$

$e^{16}=x^2-4$

$4+e^{16}=x^2$

$\pm\sqrt{4+e^{16}}=x$

$x=\sqrt{4+e^{16}}\approx 2980.959$

$\{\sqrt{4+e^{16}}\}$ or approx $\{2980.959\}$

b. $\frac{1}{30}\cdot\frac{1}{2}=30e^{3t}\cdot\frac{1}{30}$

$\ln\left(\frac{1}{60}\right)=\ln\left(e^{3t}\right)$

$\ln\left(\frac{1}{60}\right)=3t$

$\ln 1-\ln 60=3t$

$\frac{-\ln 60}{3}=t\approx -1.365$

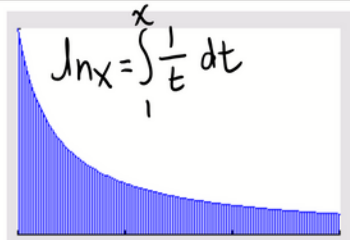
$\left\{-\frac{\ln 60}{3}\right\}$ or approx $\{-1.365\}$

DERIVATIVE OF THE NATURAL LOGARITHMIC FUNCTION

Proof:

$$\frac{d}{dx}\ln u = \frac{d}{du}\left[\int_1^u \frac{1}{t}\,dt\right]\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}$$



$x \geq 1$ so if u is a function of x, we can assume it's positive

2nd FTOC

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

if $u = f(x)$

$$\frac{d}{du}\left[\int_a^u f(t)\,dt\right]\cdot\frac{du}{dx} = f(u)$$

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad f(t) = \frac{1}{t}$$

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad f(u) = \frac{1}{u}$$

$$\frac{d}{du}\cdot\frac{du}{dx} = \frac{d}{dx}$$

[redacted]

THEOREM: DERIVATIVES OF THE NATURAL LOGARITHMIC FUNCTION

Let u be a differentiable function of x such that $u \neq 0$, then

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}$

2. $\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$

Example 4: Find the derivative with respect to x.

a. $y = \ln 5x$

$\frac{d}{dx} y = \frac{d}{dx} \ln 5x$

$y' = \frac{1}{5x} \cdot \frac{d}{dx}(5x)$

$y' = \frac{1}{5x} \cdot 5$

$y' = \frac{1}{x}$

b. $f(x) = \ln(1-2x)$

$\frac{d}{dx} f(x) = \frac{d}{dx} \ln(1-2x)$

$f'(x) = \frac{1}{1-2x} \frac{d}{dx}(1-2x)$

$f'(x) = \frac{1}{1-2x} \cdot (-2)$

$f'(x) = \frac{-2}{1-2x}$ or $f'(x) = \frac{2}{2x-1}$

c. $f(x) = \ln x^x$

$\frac{d}{dx} f(x) = \frac{d}{dx} x \ln x$

$f'(x) = \left(\frac{d}{dx} x\right) \ln x + x \frac{d}{dx}(\ln x)$

$f'(x) = 1 \ln x + x \cdot \frac{1}{x}$

$f'(x) = 1 + \ln x$

d. $y=\ln\sqrt{\dfrac{x^2+1}{1-x}}$ → $y=\frac{1}{2}\ln\frac{x^2+1}{1-x}$

$$\frac{d}{dx}y=\frac{d}{dx}\frac{1}{2}\left[\ln(x^2+1)-\ln(1-x)\right]$$

$$y'=\frac{1}{2}\left[\frac{d}{dx}\ln(x^2+1)-\frac{d}{dx}\ln(1-x)\right]$$

$$y'=\frac{1}{2}\left[\frac{\frac{d}{dx}(x^2+1)}{x^2+1}-\frac{\frac{d}{dx}(1-x)}{1-x}\right]$$

$$y'=\frac{1}{2}\left[\frac{2x}{x^2+1}-\frac{-1}{1-x}\right]$$

$$y'=\frac{1}{2}\left(\frac{2x(1-x)+1(x^2+1)}{(x^2+1)(1-x)}\right)$$

$$y'=\frac{2x-2x^2+x^2+1}{2(x^2+1)(1-x)}$$

$$y'=\frac{-x^2+2x+1}{2(x^2+1)(1-x)}$$

$$\boxed{y'=\frac{-(x^2-2x-1)}{2(x^2+1)(1-x)}}$$

e. $y=\ln\cos^2 x$ → $y=\ln(\cos x)^2$

$$\frac{d}{dx}y=\frac{d}{dx}2\ln\cos x$$

$$\frac{dy}{dx}=2\,\frac{\frac{d}{dx}\cos x}{\cos x}$$

$$\frac{dy}{dx}=\frac{2[-\sin x]}{\cos x}$$

$$\boxed{\frac{dy}{dx}=-2\tan x}$$

f. $\ln xy + x^2 - y = 10$

$$\frac{d}{dx}\left[\ln(x) + \ln(y) + x^2 - y\right] = \frac{d}{dx}10$$

$$\frac{d}{dx}\ln x + \frac{d}{dx}\ln y + \frac{d}{dx}x^2 - \frac{d}{dx}y = 0$$

$$\frac{1}{x} + \frac{\frac{d}{dx}y}{y} + 2x - \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\cdot\frac{1}{y} - \frac{dy}{dx}\cdot 1 = -\frac{1}{x} - 2x\,\frac{x}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - 1\frac{y}{y}\right) = \frac{-1-2x^2}{x}$$

$$\frac{dy}{dx}\left(\frac{1-y}{y}\right) = -\frac{(1+2x^2)}{x}$$

properties of K

$$\ln xy = \ln(xy)$$

$$\ln\frac{x}{y} = \ln\left(\frac{x}{y}\right)$$

$$\ln x + K = K + \ln x$$

$$\ln(x+K) = \ln(x+K)$$

$$\frac{d}{dx}y = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-(1+2x^2)}{(x)}\cdot\frac{(y)}{(1-y)}$$

$$\boxed{\frac{dy}{dx} = \frac{y(1+2x^2)}{x(y-1)}}$$

THEOREM: DERIVATIVE INVOLVING ABSOLUTE VALUE

Let u be a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}\left[\ln|u|\right] = \frac{1}{u}\cdot\frac{du}{dx}$$

Proof:

$$|u| = \begin{cases} -u, & u < 0 \\ u, & u > 0 \end{cases}$$

Case 1: let $u > 0$.

$$\frac{d}{dx}\ln|u| = \frac{d}{dx}\ln(u) = \frac{1}{u}\cdot\frac{du}{dx} \checkmark$$

Case 2: let $u < 0$

$$\frac{d}{dx}\ln|u| = \frac{d}{dx}\ln(-u) = \frac{1}{-u}\cdot\frac{d}{dx}(-u) = \frac{-\frac{du}{dx}}{-u} = \frac{1}{u}\cdot\frac{du}{dx} \checkmark$$

Example 5: Find the derivative of the function with respect to *x*.

$\frac{d}{dx} f(x) = \frac{d}{dx} \ln|\sec x + \tan x|$

$\frac{d}{dx} \ln|u| = \frac{1}{u} \cdot \frac{du}{dx}$

$f'(x) = \frac{\frac{d}{dx}(\sec x + \tan x)}{\sec x + \tan x}$

$f'(x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$(mn)^{1/2} = m^{1/2} n^{1/2}$

$(m^2 n q)^{1/2} = m^{2(1/2)} n^{1/2} q^{1/2}$

Logarithmic Differentiation

Example 6: Differentiate the following functions with respect to *x*.

a. $f(x) = \sqrt{x^2(x+1)(x+2)}$

$\ln y = \ln\left[x^2(x+1)(x+2)\right]^{1/2}$

$\frac{d}{dx} \ln y = \frac{d}{dx} \frac{1}{2}\left[\ln x^2 + \ln(x+1) + \ln(x+2)\right]$

$y \cdot \frac{y'}{y} = y \cdot \frac{1}{2}\left[\frac{2x}{x^2} + \frac{1}{x+1} + \frac{1}{x+2}\right]$

$y' = \frac{y}{2}\left[\frac{2(x+1)(x+2) + 1(x)(x+2) + 1(x)(x+1)}{x(x+1)(x+2)}\right]$ with x^2+3x+2, x^2+2x, x^2+x

$y' = \frac{y}{2}\left[\frac{2x^2+6x+4 + x^2+2x + x^2+x}{x(x+1)(x+2)}\right]$

$y' = \frac{x(x+1)^{1/2}(x+2)^{1/2}}{2}\left[\frac{4x^2+9x+4}{x(x+1)^1(x+2)^1}\right]$

$y' = \frac{4x^2+9x+4}{2(x+1)^{1/2}(x+2)^{1/2}}$

$y' = \frac{4x^2+9x+4}{2\sqrt{(x+1)(x+2)}}$

b. $y=\left(\frac{x^2-4}{x^2+4}\right)^{2/3}$ → $y=\frac{(x^2-4)^{2/3}}{(x^2+4)^{2/3}}$ for later

$$\ln y=\ln\left(\frac{x^2-4}{x^2+4}\right)^{2/3}$$

$$\frac{d}{dx}\ln y=\frac{d}{dx}\frac{2}{3}\left[\ln(x^2-4)-\ln(x^2+4)\right]$$

$$y\cdot\frac{y'}{y}=y\cdot\frac{2}{3}\left[\frac{2x}{x^2-4}-\frac{2x}{x^2+4}\right]$$

$$y'=\frac{2y}{3}\left[\frac{2x(x^2+4)-2x(x^2-4)}{(x^2-4)(x^2+4)}\right]$$

$$y'=\frac{2}{3}\cdot\frac{(x^2-4)^{2/3}}{(x^2+4)^{2/3}}\left[\frac{2x^3+8x-2x^3+8x}{(x^2-4)(x^2+4)}\right]$$

$$y'=\frac{2(16x)}{3(x^2-4)^{1/3}(x^2+4)^{5/3}}$$

$$\boxed{y'=\frac{32x}{3\sqrt[3]{(x^2-4)(x^2+4)^5}}}$$

Example 7: Find an equation of the tangent line of the function $f(x)=\sin 2x \ln x^2$ at the point $(1,0)$. $\Rightarrow (1, f(1)) \rightarrow f(1)=0$

$$f(x)=(\sin 2x)(2\ln x)$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[2(\sin 2x)(\ln x)\right]$$

$$f'(x)=2\left[2(\cos 2x)(\ln x)+(\sin 2x)\left(\frac{1}{x}\right)\right]$$

$$f'(1)=2\left[2\cos[2(1)][\ln(1)]+[\sin 2(1)]\left(\frac{1}{1}\right)\right]$$

$$f'(1)=2[0+\sin 2]$$

$f'(1)=2\sin 2 \rightarrow (1, 2\sin 2)$ is on the graph of $f'(x)$

Point-slope form of a line

$$y-y_1=m(x-x_1)$$

$$y-0=2\sin 2(x-1)$$

$$y=(2\sin 2)(x-1)$$

5.2: The Natural Logarithmic Function: Integration

When you are done with your homework you should be able to…

- π Use the Log Rule for Integration to integrate a rational function
- π Integrate trigonometric functions

Warm-up:

1. Differentiate the following functions with respect to *x*.

a. $y = x\ln 5x$

$\frac{d}{dx}y = \frac{d}{dx}(x\ln 5x)$

$y' = \left(\frac{d}{dx}x\right)(\ln 5x) + x\left(\frac{d}{dx}\ln 5x\right)$

$y' = 1\ln 5x + x\left(\frac{5}{5x}\right)$

$y' = 1 + \ln 5x$

b. $\ln(xy) = \ln(x+y)$.

$\frac{d}{dx}(xy) = \frac{d}{dx}(x+y)$

$\left(\frac{d}{dx}x\right)y + x\left(\frac{d}{dx}y\right) = \left(\frac{d}{dx}x\right) + \left(\frac{d}{dx}y\right)$

$1y + x\cdot y' = 1 + y'$

$xy' - y' = 1 - y$

$y'(x-1) = 1-y$

$y' = \frac{1-y}{x-1}$

$\ln xy = \ln(x+y)$

$\frac{d}{dx}\ln x + \frac{d}{dx}\ln y = \frac{d}{dx}\ln(x+y)$

$\frac{1}{x} + \frac{y'}{y} = \frac{\frac{d}{dx}x + \frac{d}{dx}y}{x+y}$

$xy(x+y)\left(\frac{1}{x} + \frac{y'}{y}\right) = \left(\frac{1+y'}{x+y}\right)xy(x+y)$

$y(x+y) + x(x+y)y' = xy + xyy'$

$xy + y^2 + x^2y' + xyy' = xy + xyy'$

$x^2y' = -y^2$

$y' = -\frac{y^2}{x^2}$

"This is correct… I'm not sure what happened on the other side"

THEOREM: LOG RULE FOR INTEGRATION

Let u be a differentiable function of x.

$\int \frac{1}{u} du = \ln|u| + C$

1. $\int \frac{1}{x} dx = \ln|x| + C$

2. $\int \frac{u'}{u} dx = \ln|u| + C$

$\int \frac{1}{x} dx = \int \frac{dx}{x} = \int x^{-1} dx$

$\int \frac{1}{u} du = \int \frac{du}{u} = \int u^{-1} du$

$\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

Example 1: Find or evaluate the integral.

a. $\int \frac{10}{x} dx = 10 \int \frac{1}{x} dx$

$= 10 \int x^{-1} dx$

$= 10 \ln|x| + C$

$= \ln|x|^{10} + C$

$= \boxed{\ln x^{10} + C}$

b. $\int \frac{x^2}{\sqrt{5-x^3}}dx$ $= \int (x^2)(5-x^3)^{-1/2}\,dx$

$u = 5 - x^3$

$\frac{du}{dx} = -3x^2$

$dx = \frac{du}{-3x^2}$

$= \int x^2 u^{-1/2} \left(\frac{du}{-3x^2}\right)$

$= -\frac{1}{3}\int u^{-1/2}\,du$

$= -\frac{1}{3}\frac{u^{-1/2+1}}{-\frac{1}{2}+1} + C$

$= -\frac{1}{3}\frac{u^{1/2}}{1/2} + C$

$= -\frac{2}{3}(5-x^3)^{1/2} + C$

c. $\int \frac{x}{\sqrt{1-x^2}}dx$ $= \int x(1-x^2)^{-1/2}\,dx$

$u = 1 - x^2$

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$

$= \int x u^{-1/2}\left(\frac{du}{-2x}\right)$

$= -\frac{1}{2}\int u^{-1/2}\,du$

$= -\frac{1}{2}\frac{u^{1/2}}{1/2} + C$

$= -(1-x^2)^{1/2} + C$

d. $\int_e^{e^2} \frac{dx}{x\ln x} = \int_e^{e^2} x^{-1}(\ln x)^{-1}\,dx$

$= \int_1^2 x^{-1} u^{-1} (x\,du)$

$x^{-1}\cdot x^{1} = x^{-1+1} = x^0 = 1$

$= \int_1^2 u^{-1}\,du$

$= \ln|u| \Big|_{u=1}^{u=2}$

$= \ln|2| - \ln|1|$ (with $\ln|1| = 0$)

$= \boxed{\ln 2}$

$(\ln x)^2 = (\ln x)(\ln x)$

vs

$\ln x^2 = 2\ln x$

$u = \ln x$ or $u(x) = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

upper lim: $u(e^2) = \ln e^2 = 2$

lower lim: $u(e) = \ln e = 1$

e. $\int_1^e \frac{(1+\ln x)^2\,dx}{x}$

$= \int_1^2 \frac{u^2 (x\,du)}{x}$

$\int_1^e \frac{(1+\ln x)\,dx}{x} \neq \int_1^e u^2\,du$

$u = u(x) = 1 + \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

upper lim: $u(e) = 1 + \ln e = 2$

lower lim: $u(1) = 1 + \ln 1 = 1$

$= \int_1^2 u^2\,du$

$= \frac{1}{3}u^3 \Big|_{u=1}^{u=2}$

$\int \frac{(1+\ln x)^2\,dx}{x} = \int u^2\,du$

$= \frac{1}{3}u^3$

$= \frac{1}{3}(1+\ln x)^3$

$= \frac{1}{3}(8-1)$

$= \boxed{\frac{7}{3}}$

So $\int_1^e \frac{(1+\ln x)\,dx}{x} = \frac{1}{3}(1+\ln x)^3 \Big|_{x=1}^{x=e}$

f. $\int \frac{1}{x^{2/3}\left(1+x^{1/3}\right)}dx = \int x^{-2/3}\left(1+x^{1/3}\right)^{-1}dx$

$u = 1+x^{1/3}$

$\frac{du}{dx} = \frac{1}{3}x^{-2/3}$

$dx = \frac{3\,du}{x^{-2/3}}$

$= \int x^{-2/3}u^{-1}\left(\frac{3\,du}{x^{-2/3}}\right)$

$= 3\int u^{-1}\,du$

$= 3\ln|u| + C$

$= \ln|u|^3 + C$

$= \ln\left|1+x^{1/3}\right|^3 + C$

g. $\int \frac{x^3-6x-20}{x+5}dx$

$$\begin{array}{r} x^2-5x+19+\frac{-115}{x+5} \\ (x+5)\overline{)\,x^3+0x^2-6x-20} \\ -(x^3+5x^2) \\ \hline -5x^2-6x \\ -(-5x^2-25x) \\ \hline 19x-20 \\ -(19x+95) \\ \hline -115 \end{array}$$

$= \int\left(x^2-5x+19+\frac{-115}{x+5}\right)dx$

$= \int(x^2-5x+19)dx - 115\int(x+5)^{-1}dx$

$u = x+5$

$\frac{du}{dx} = 1$

$dx = du$

$= \frac{1}{3}x^3 - \frac{5}{2}x^2 + 19x - 115\int u^{-1}du$

$= \frac{1}{3}x^3 - \frac{5}{2}x^2 + 19x - 115\ln|u| + C$

$= \boxed{\frac{1}{3}x^3 - \frac{5}{2}x^2 + 19x - 115\ln|x+5| + C}$

h. $\int \tan\theta\, d\theta = \int \frac{\sin\theta}{\cos\theta}\, d\theta$

$u = \cos\theta$

$\frac{du}{d\theta} = -\sin\theta$

$d\theta = \frac{du}{-\sin\theta}$

$= \int \frac{\sin\theta}{u} \cdot \frac{du}{-\sin\theta}$

$= -\int \frac{du}{u}$

$= -\ln|u| + C$

$= \boxed{-\ln|\cos\theta| + C}$

i. $\int \cot\theta\, d\theta = \int \frac{\cos\theta}{\sin\theta}\, d\theta$

$u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$d\theta = \frac{du}{\cos\theta}$

$= \int \frac{\cos\theta}{u} \cdot \frac{du}{\cos\theta}$

$= \int \frac{1}{u}\, du$

$= \ln|u| + C$

$= \boxed{\ln|\sin\theta| + C}$

$\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$

$\frac{d}{d\theta}\tan\theta = \sec^2\theta$

j. $\int \sec\theta d\theta \frac{(\tan\theta + \sec\theta)}{(\tan\theta + \sec\theta)}$

$u = \tan\theta + \sec\theta$

$\frac{du}{d\theta} = \sec^2\theta + \sec\theta\tan\theta$

$d\theta = \frac{du}{\sec^2\theta + \sec\theta\tan\theta}$

$= \int \frac{(\sec\theta\tan\theta + \sec^2\theta)d\theta}{\tan\theta + \sec\theta}$

$= \int \frac{(\sec\theta\tan\theta + \sec^2\theta)}{u} \cdot \frac{du}{\sec^2\theta + \sec\theta\tan\theta}$

$= \int \frac{du}{u}$

$= \ln|u| + C$

$= \boxed{\ln|\sec\theta + \tan\theta| + C}$

k. $\int \csc\theta d\theta \frac{(\csc\theta + \cot\theta)}{(\csc\theta + \cot\theta)}$

$u = \csc\theta + \cot\theta$

$\frac{du}{d\theta} = -\csc\theta\cot\theta - \csc^2\theta$

$d\theta = \frac{du}{-(\csc\theta\cot\theta + \csc^2\theta)}$

$= \int \frac{(\csc^2\theta + \csc\theta\cot\theta)d\theta}{\csc\theta + \cot\theta}$

$= \int \frac{\csc^2\theta + \csc\theta\cot\theta}{u} \cdot \frac{du}{-(\csc\theta\cot\theta + \csc^2\theta)}$

$= -\int \frac{1}{u}du \longrightarrow = -\ln|u| + C \rightarrow = \boxed{-\ln|\csc\theta + \cot\theta| + C}$

INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$\int \sin u\,du = -\cos u + C$	$\int \cos u\,du = \sin u + C$
$\int \tan u\,du = -\ln\lvert\cos u\rvert + C$	$\int \cot u\,du = \ln\lvert\sin u\rvert + C$
$\int \sec u\,du = \ln\lvert\sec u + \tan u\rvert + C$	$\int \csc u\,du = -\ln\lvert\csc u + \cot u\rvert + C$

Example 2: Solve the differential equation.

a. $y' = \dfrac{x+1}{x-1}$

$$dx\,\frac{dy}{dx} = \frac{x+1}{x-1}\,dx$$

$$\int 1\,dy = \int \frac{x+1}{x-1}\,dx$$

$$y = \int \left(1 + \frac{2}{x-1}\right)dx$$

$$y = x + 2\ln|x-1| + C$$

$$y = x + \ln(x-1)^2 + C$$

$$(x-1)\overline{)x+1} = 1 + \frac{2}{x-1}, \quad x+1-(x-1) = 2$$

$$\int (x+1)(x-1)^{-1}\,dx$$

$$= \int (x+1)u^{-1}\,du$$

$$= \int (u+1+1)u^{-1}\,du$$

$$= \int (u+2)u^{-1}\,du$$

$$= \int (1+2u^{-1})\,du$$

$$= u + 2\ln|u| + C_1$$

$$= x-1 + 2\ln|x-1| + C_1$$

$$= x + \ln(x-1)^2 + C$$

$u = x-1$, $\dfrac{du}{dx} = 1$, $x = u+1$

b. $r' = \theta \tan \theta^2$

$$\frac{dr}{d\theta} = \theta \tan\theta^2$$

$$\int dr = \int \theta \tan\theta^2\,d\theta$$

$$r = \int \theta \tan u \left(\frac{du}{2\theta}\right)$$

$$r = \frac{1}{2}\int \tan u\,du$$

$u = \theta^2$, $\dfrac{du}{d\theta} = 2\theta$, $d\theta = \dfrac{du}{2\theta}$

$$r = \frac{1}{2}\left[-\ln|\cos u|\right] + C$$

$$r = -\frac{1}{2}\ln|\cos\theta^2| + C$$

Example 3: The demand equation for a product is $p = \frac{90,000}{400+3x}$. Find the average price on the interval $40 \le x \le 50$.

$$\text{Average price} = \frac{1}{b-a}\int_a^b p(x)\,dx$$

$u = 400 + 3x$

$\frac{du}{dx} = 3$

$dx = \frac{du}{3}$

upper lim:

$u(50) = 400 + 3 \cdot 50 = 550$

$u(40) = 400 + 3 \cdot 40 = 520$

$$= \frac{1}{50-40}\int_{p=40}^{p=50} 90000(400+3x)^{-1}\,dx$$

$$= 9000\int_{u=520}^{u=550} u^{-1}\,\frac{du}{3}$$

$$= 3000 \ln|u| \Big|_{u=520}^{u=550}$$

$$= 3000(\ln 550 - \ln 520)$$

$$= 3000 \ln \frac{550}{520}$$

$$= 3000 \ln \frac{55}{52}$$

$$= \boxed{\$168.27}$$

$= \text{average price}$

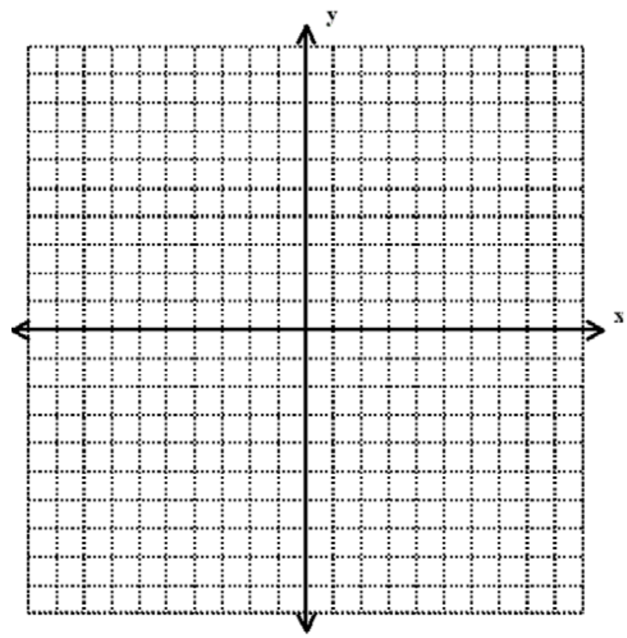
5.3: Inverse Functions

When you are done with your homework you should be able to…

- π Verify that one function is the inverse of another function
- π Determine whether a function has an inverse function
- π Find the derivative of an inverse function

Warm-Up:

1. Use the Horizontal Line Test to show that $f(x) = x^5 + 3$ is one-to-one.



2. Let $g(x)=\frac{1-2x}{x}$ and $h(x)=\frac{x}{x-2}$ find $g(h(x))$

Recall from Precalculus:

Definition of Inverse Function

A function g is the **inverse function** of the function f when

$f(g(x)) = x$ for each x in the domain of g

and

$g(f(x)) = x$ for each x in the domain of f.

The function g is denoted by f^{-1} (read "f inverse").

Important Observations about inverse functions:

1. If g is the ___________________ function of f, then ______ is the inverse function of _______.
2. The ______________ of f^{-1} is equal to the _______________ of _____, and the _____________ of f^{-1} is equal to the _________________ of _____.
3. If a function has an inverse, the inverse is ________________.
4. $f^{-1}(f(x)) =$ __________ and $f(f^{-1}(x)) =$ ______________.
5. A function has an inverse if and only if it is ___________ - ______ -_________.

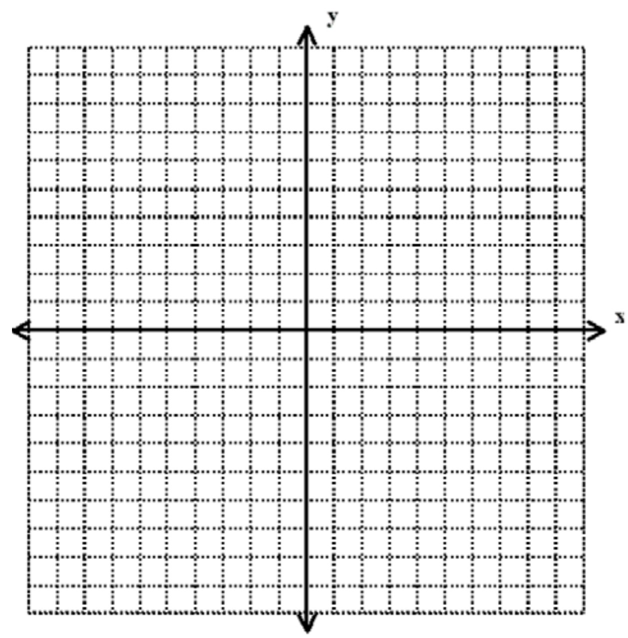
THEOREM 5.7 The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

Proof:

Example 1: Graph each function and verify that the following functions are inverses of each other.

$f(x)=16-x^2,\ x\leq 0 \quad g(x)=-\sqrt{16-x}$



GUIDELINES FOR FINDING AN INVERSE FUNCTION

1. Use Theorem 5.7 to determine whether the function $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Example 2: Find f^{-1} of $f(x)=\dfrac{3x+2}{1-x}$.

Example 3: Prove that $f(x)=x^3+2x-3$ has an inverse function (you do not need to find it).

Derivative of an Inverse Function:

THEOREM 5.8 Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof:

THEOREM 5.9 **The Derivative of an Inverse Function**

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof:

Example 4. Let $f(x) = x^3 + 2x - 3$

a. What is the value of $f^{-1}(x)$ when x = 0?

b. What is the value of $f'(x)$ when x = 0?

c. What is the value of $(f^{-1})'(x)$ when x = 0?

d. What do you notice about the slopes of f and f^{-1}

Graphs of inverse functions have ____________________________ slopes.

5.4: Exponential Functions: Differentiation and Integration

When you are done with your homework you should be able to…

- π Develop properties of the natural exponential function
- π Differentiate natural exponential functions
- π Integrate natural exponential functions

Warm-up:

1. Differentiate the following functions with respect to *x*.

a. $y = x^{5x}$

b. $f\left(x\right) = \ln e^{\cos 2x}$.

DEFINITION: THE NATURAL EXPONENTIAL FUNCTION

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

That is,

The inverse relationship between the natural logarithmic function and the natural exponential function can summarized as follows:

Example 1: Solve the following equations. Give the **exact result** and then round to 3 decimal places.

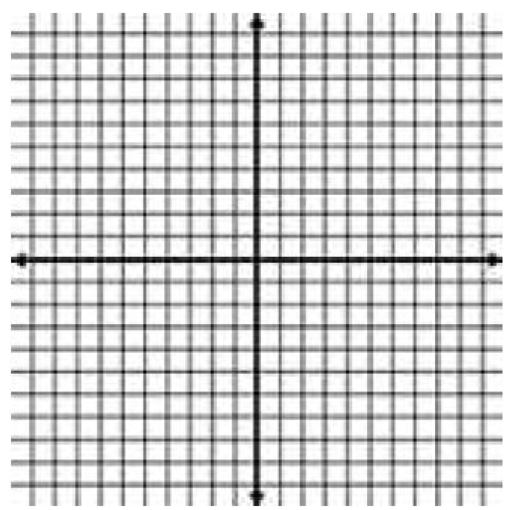
c. $e^{\ln 6x} = 20$

d. $\dfrac{5000}{1+e^{2x}}=2$

e. $\ln 8x=3$

f. $e^{2x} - 2e^{x} - 8 = 0$

Example 2: Sketch the graph of $f(x) = 2e^{x-1}$ without using your graphing calculator.



DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

THEOREM: DERIVATIVES OF THE NATURAL EXPONENTIAL FUNCTION

Let u be a differentiable function of x.

1. $\frac{d}{dx}\left[e^x\right]=e^x$

2. $\frac{d}{dx}\left[e^u\right]=$ ____________________

Example 3: Find the derivative with respect to x.

g. $y=e^{5-x^3}$

h. $f\left(x\right)=xe^{3x}$

i. $y=\ln\frac{1+e^x}{1-e^x}$

j. $y=\frac{e^{x}-e^{-x}}{2}$

k. $e^{xy}+x^2-y^2=10$

l. $F(x)=\int_0^{e^{2x}} \ln(t+1)dt$

Example 4: Find an equation of the tangent line of the function $1+\ln xy = e^{x-y}$ at the point $(1,1)$.

Example 5: Find the extrema and points of inflection of the function

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-3)^2/2}.$$

THEOREM: INTEGRATION RULES FOR NATURAL EXPONENTIAL FUNCTIONS

Let u be a differentiable function of x.

1. $\int e^x dx =$ ______________
2. $\int e^u du =$ ______________

Example 6: Find the indefinite integrals and evaluate the definite integrals.

a. $\int e^{12x} dx$

b. $\int x^4 e^{1-x^5} dx$

c. $\int \frac{e^{2x}}{1+e^{2x}} dx$

d. $\int \frac{e^{2x}+2e^{x}+1}{e^{x}} dx$

e. $\int e^{\tan 2x} \sec^2 2x dx$

f. $\int_0^1 \frac{e^x}{5-e^x} dx$

Example 7: Solve the differential equation.

$$\frac{dy}{dx} = \left(e^{x} - e^{-x}\right)^{2} dx$$

Example 8: The median waiting time (in minutes) for people waiting for service in a convenience store is given by the solution of the equation $\int_0^x 0.3e^{-0.3t}\,dt = \frac{1}{2}$. Solve the equation.